

# TRB Annual Meeting

## Traffic Assignment of Multi-Modal Transport Networks Based on An Augmented Link-Based Super-Network Model --Manuscript Draft--

<b>Full Title:</b>	Traffic Assignment of Multi-Modal Transport Networks Based on An Augmented Link-Based Super-Network Model
<b>Abstract:</b>	<p>Multi-modal transport is getting more popular due to the emergence of new traffic modes. The increase of modes also adds complexity for the transport researchers. This paper proposes an augmented link-based super-network approach for modeling multi-modal transport networks, addressing the scalability and versatility issues of conventional methods. This approach is used to calculate the user equilibrium for urban transport networks traffic assignment with multiple traffic modes, a difficult problem due to the intractable enumeration of feasible paths between origin-destination pairs and restricted transfers between different traffic modes. In the super-network representation of multi-modal transport networks, the travel cost of any feasible route between the origin and destination is formulated as the sum of cost functions of the augmented links, thus avoiding the enumeration of feasible paths. Additionally, restrictions on traffic mode transfers can be embedded in the link-based model by excluding infeasible transfer links or adding penalties for undesired transfers. The user equilibrium of the augmented link-based super-network model is formulated as a variational inequality problem, solved using the extra-gradient algorithm. A multi-modal transport network is considered in the case study. Simulation results validate the effectiveness of the proposed model, demonstrating its scalability and versatility in addressing complex multi-modal transport networks with diverse traffic modes. We anticipate that our method can serve as an efficient modeling approach for more general and complex multi-modal transport networks, facilitating traffic management and network design.</p>
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1 **TRAFFIC ASSIGNMENT OF MULTI-MODAL TRANSPORT NETWORKS BASED ON**  
2 **AN AUGMENTED LINK-BASED SUPER-NETWORK MODEL**

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**1 ABSTRACT**

2 Multi-modal transport is getting more popular due to the emergence of new traffic modes. The  
3 increase of modes also adds complexity for the transport researchers. This paper proposes an  
4 augmented link-based super-network approach for modeling multi-modal transport networks, ad-  
5 dressing the scalability and versatility issues of conventional methods. This approach is used to  
6 calculate the user equilibrium for urban transport networks traffic assignment with multiple traffic  
7 modes, a difficult problem due to the intractable enumeration of feasible paths between origin-  
8 destination pairs and restricted transfers between different traffic modes. In the super-network  
9 representation of multi-modal transport networks, the travel cost of any feasible route between  
10 the origin and destination is formulated as the sum of cost functions of the augmented links, thus  
11 avoiding the enumeration of feasible paths. Additionally, restrictions on traffic mode transfers can  
12 be embedded in the link-based model by excluding infeasible transfer links or adding penalties  
13 for undesired transfers. The user equilibrium of the augmented link-based super-network model  
14 is formulated as a variational inequality problem, solved using the extra-gradient algorithm. A  
15 multi-modal transport network is considered in the case study. Simulation results validate the ef-  
16 fectiveness of the proposed model, demonstrating its scalability and versatility in addressing com-  
17 plex multi-modal transport networks with diverse traffic modes. We anticipate that our method  
18 can serve as an efficient modeling approach for more general and complex multi-modal transport  
19 networks, facilitating traffic management and network design.

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21 *Keywords:* multi-modal traffic, traffic assignment, user equilibrium, augmented link-based model

## 1 INTRODUCTION

2 Multi-modal transport networks consist of various traffic modes, including private cars, public  
3 transit, active mobility, shared mobility, and more. Passengers can use a single mode or a combi-  
4 nation of multiple modes, transferring between different modes to complete a trip. However, with  
5 ongoing urbanization and the growing size of cities, it becomes increasingly difficult to complete  
6 a trip using a single mode. From an individual traveler's perspective, the emergence of new traffic  
7 modes offers more choices for their daily commute and can meet the diverse travel demands of  
8 different types of passengers. From a collective perspective, multi-modal transport can facilitate  
9 achieving seamless travel and maximizing traffic capacity at the network level.

10 User equilibrium (UE) is a traffic flow assignment model, which has received considerable  
11 attention, since it is essential for managing and designing transport networks as well as evaluating  
12 traffic system performance. In a user equilibrium, all the users choose their best routes selfishly.  
13 An equilibrium is reached eventually, where the users sharing the same origin and destination will  
14 experience the same travel cost, irrespective of the chosen route ( $I$ ). Note that "cost" includes  
15 monetary costs, as well as other disutilities (travel time, discomfort, etc.). In this study, the UE  
16 result is used to estimate the behavior of users with given demands and network settings, and  
17 determine the flows on each link and the optimal routes in multi-modal transport networks.

18 Adequate traffic network supply modelling is essential to estimating the collective be-  
19 haviour of users, and thereby the impact that different measures, e.g. supply management or  
20 pricing, might have on traffic performance. There are extensive researches on the modelling of  
21 traffic network supply. One of the earliest examples is the Bureau of Public Roads (BPR) func-  
22 tion (2), which estimates the travel cost of road links based on traffic flows. BPR and BPR-type  
23 functions have been widely used to approximate UE in traffic assignment processes, for instance  
24 in (3–6). BPR functions have not only been employed for estimating the travel time on road links,  
25 but also for approximating the waiting time for public transit in multi-modal transport networks.  
26 Other methods for modelling multi-modal traffic networks also exist. Pi et al. (7) studied a general  
27 formulation for multi-modal dynamic traffic assignment considering multi-class vehicles, public  
28 transit and parking. In this formulation, the dynamics of each traffic mode and the interactions  
29 among different modes are explicitly considered, which allows for the calculation for dynamic  
30 user equilibrium and the study of passenger route choices among different modes. Lo et al. (3)  
31 explicitly considered the restrictions on transfers among different modes in multi-modal transport  
32 networks. They proposed a state-augmented multi-modal (SAM) network model to encode the  
33 probable transfer rules as well as the maximum number of transfers. In addition, they constructed  
34 a direct in-vehicle link between each pair of connected nodes in each modal sub-network, in order  
35 to address non-linear and non-additive transit fares.

36 However, most of the existing works on multi-modal transport modelling relies on calcu-  
37 lation of path travel costs in their assignment procedure, which means the enumeration of feasible  
38 paths for any OD pair is necessary in order to calculate UE. This is mainly due to the presence  
39 of travel cost components which are dependent on the travel information of the entire path (that  
40 is, path-additive) (8). For example, the restrictions on feasible transfers among different modes or,  
41 similarly, behavioural considerations w.r.t. the maximum number of acceptable transfers, represent  
42 non-separable path-based cost dynamics.

43 In large scale networks, this will result in combinatorial explosion of the number of possible  
44 paths, severely impacting computation times for enumeration schemes. Thus the scalability of  
45 path-based modelling methods can be significantly limited. Szeto and Jiang (9) proposed a link-

1 based method to address the scalability issue of traffic assignment. But the method is dedicated to  
2 transit assignment. *To the best of our knowledge, there is not an efficient and general modelling*  
3 *method for multi-modal transport networks yet.*

4 Therefore, in this work we propose an augmented link-based super-network model to ad-  
5 dress this issue. Through employing the super-network framework and the augmented link concept,  
6 the path costs can be formulated as a sum of the augmented link costs. In addition, the proposed  
7 method can encode the feasible transfers among different traffic modes. Thus the enumeration of  
8 feasible paths is avoided. the method is not only scalable to larger traffic networks, but also can  
9 accommodate a wide range of traffic modes. The contributions of the paper include the following:

- 10 • Formulating a novel augmented link-based super-network model for multi-modal trans-  
11 port networks, which is scalable and versatile to complex networks and diverse traffic  
12 modes.
- 13 • Formulating the UE problem of the augmented link-based super-network model as a  
14 variational inequality problem, and providing a solution for the generated variational  
15 inequality problem based on an extra-gradient method.
- 16 • Validating the effectiveness of the proposed method and showing the equivalence to the  
17 conventional path-based method through a case study; showing the versatility of the  
18 method by considering diverse traffic modes in the case study.

19 The rest of this paper is organized as follows. In Section 3, the augmented link-based super-  
20 network model is presented to formulate the multi-modal transport network with an illustrative  
21 example, and the UE is formulated as a variational inequality problem which is solved with the  
22 extra-gradient algorithm. A case study is conducted based on a multi-modal transport network in  
23 Section 4 to show the efficiency and versatility of the proposed method. Finally, in Section 5 we  
24 draw some concluding remarks.

## 25 **METHODOLOGY AND FORMULATION**

26 This section will give the presentation of multi-modal traffic networks, and introduce the notation  
27 system used to formulate the model. After that, the augmented link-based super-network model is  
28 presented, and the assumptions used throughout the paper are given.

### 29 **Multi-modal transport super-network**

30 We aim to consider a general multi-modal traffic network that can accommodate a wide range of  
31 traffic modes, in which passengers can board on and alight from public transit and transfer among  
32 different modes. A widely-used expression of transport network is the graph network that consists  
33 of links and nodes, in which the nodes represent origin nodes, destination nodes, and activity nodes  
34 where passengers can park cars or transfer between modes, while the links represent the physical  
35 traffic links connecting the nodes. An illustrative example is shown in Figure 1, in which there are  
36 two origin nodes  $O_1$ ,  $O_2$  and one destination node  $D$ , and two intermediate nodes  $A$  and  $B$  where  
37 passengers can park their private cars. In addition to the road network, there are three public transit  
38 modes, i.e., bus, tram, and metro, which occupy different links of the network. In this multi-modal  
39 network, passengers can choose one single or multiple modes to complete their trips. For instance,  
40 one can take a bus or tram to travel directly from  $O_1$  to  $D$ , or park and transfer to bus or tram at the  
41 intermediate node.

42 However, the transfers between modes cannot be generally modelled through a graph  
43 whose topology is a 1:1 representation of physical topology. For example, some temporal-related

1 transfers cannot be captured by a physical link. A super-network (also known as hyper-network)  
 2 is a generalization of a conventional graph, where additional links and nodes are employed to  
 3 represent relationships between portions of the original graph that are not physical in nature, but  
 4 rather behavioral. In our adopted approach, we consider a concatenation of single modal networks  
 5 interconnected by transfer links that capture the mode transfer behaviours, through which all the  
 6 feasible transfers can be explicitly modelled (10, 11).

7 A super-network description of the multi-modal network is given in Figure 2, where all  
 8 the transfer links are explicitly depicted that describe the transfer relationships and restrictions  
 9 between modes. For example, it is possible to transfer between bus and tram at the intermediate  
 10 node bidirectionally; however this cannot be the case when transferring from private car to public  
 11 transport. Carlier et al. (8) used a super-network to model a multi-modal transport network and  
 12 embedded the transfer costs in the transfer links. However, in this approach the travel cost is still  
 13 estimated based on path route.

14 The path-dependent transfer cost also requires the path information to estimate travel costs.  
 15 For instance, the egress time of link  $O_1A$  on private car sub-network depends on the downstream  
 16 link which is path information. Similarly, the last-mile travel cost from parking node  $D$  to the  
 17 real activity position depends on the upstream link, i.e., the costs are very different depending on  
 18 whether the user's previous traffic mode is private car or transit. Furthermore, the waiting time for  
 19 public transit at one node can be influenced by the flow on the upstream link. The parking cost at  
 20 one node depends on the car flow entering the parking lot instead of the arriving car flow. This is  
 21 why path enumeration is necessary in conventional modelling methods.

22 In the next section, based on the super-network description of multi-modal network, an aug-  
 23 mented link-based super-network model is developed to address the scalability issue by modelling  
 the travel costs with augmented link cost functions.

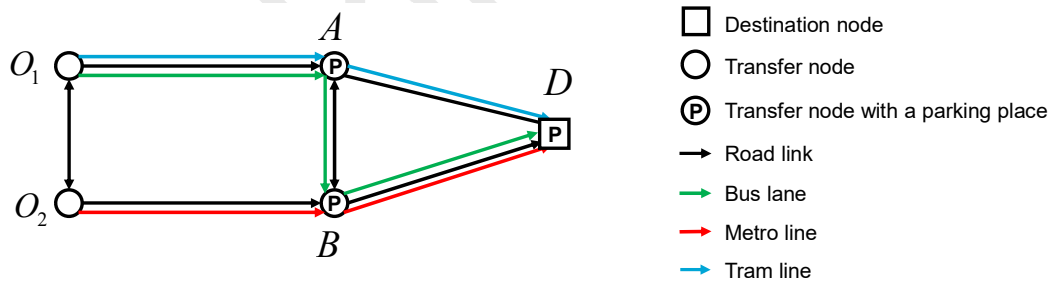
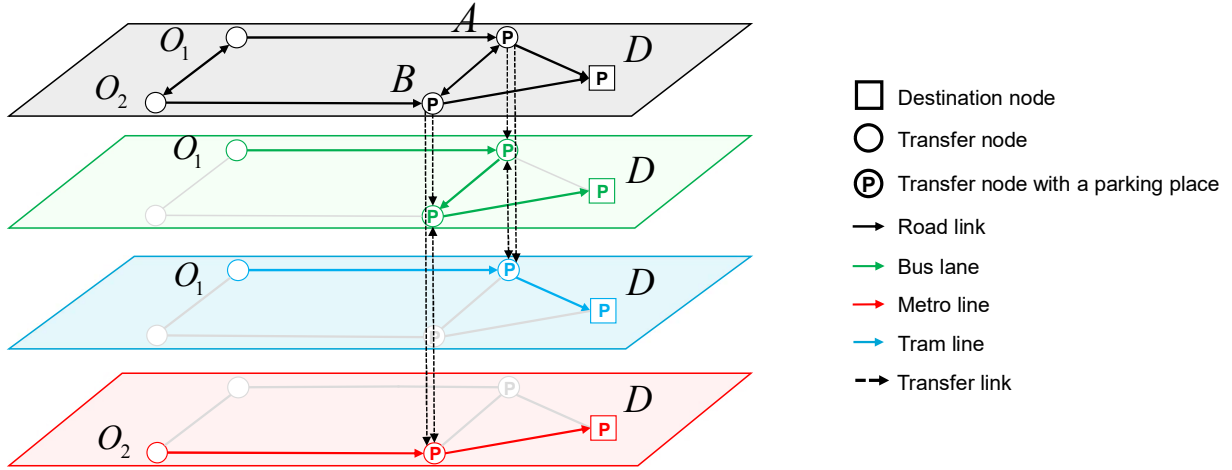


FIGURE 1: Illustrative multi-modal transport network

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### 25 **Augmented link-based super-network model**

26 An augmented link-based super-network graph of a multi-modal transport network is shown in  
 27 Figure 3. In this graph, each layer represents a single traffic mode network, in which all the mode-  
 28 specific nodes are defined accordingly. All the layers share the same origin and destination nodes.  
 29 We further extend the concept of conventional transfer links that connect two mode by defining  
 30 the *augmented transfer link*. For example, the links connecting the origin/destination nodes and  
 31 the traffic nodes belong to the augmented transfer link set, too. Therefore, in this model, links  
 32 are categorized in two types: in-vehicle links and augmented transfer links. The in-vehicle links  
 33 refer to the links connecting two nodes in the same sub-network; the augmented transfer links



**FIGURE 2:** Super-network of a multi-modal transport network

1 include the links connect two nodes from different sub-networks and the links that connect the  
 2 origin/destination nodes with from a traffic sub-network.

3 Compared to conventional multi-modal transport models, the proposed augmented link-  
 4 based super-network model computes the total route travel cost as sum of augmented-link travel  
 5 costs, without the need of the entire path information, under adequate regularity assumptions. This  
 6 is achieved by encoding all the path-dependent travel cost in the augmented transfer link cost  
 7 functions, while the in-vehicle links of each mode model the travel costs that are independent of  
 8 paths, including link travel time, the fuel cost on private car links, and distance-related transit fares.  
 9 Each type of link cost captures certain travel costs, which is independent of path. The detailed link  
 10 cost functions of each link type are defined in next subsection.

### 11 Link cost functions

12 In order to formulate the travel cost functions, the notations used throughout the paper are first  
 13 introduced in Table 1. Note that, in this paper, the time cost (i.e., time multiplied with a value of  
 14 time) and monetary cost of users are considered, and the sum of both costs is taken as the perceived  
 15 cost of users, given as:

$$16 C_a(\mathbf{f}) = \omega_t t_a(\mathbf{f}) + \omega_c c_a(\mathbf{f}). \quad (1)$$

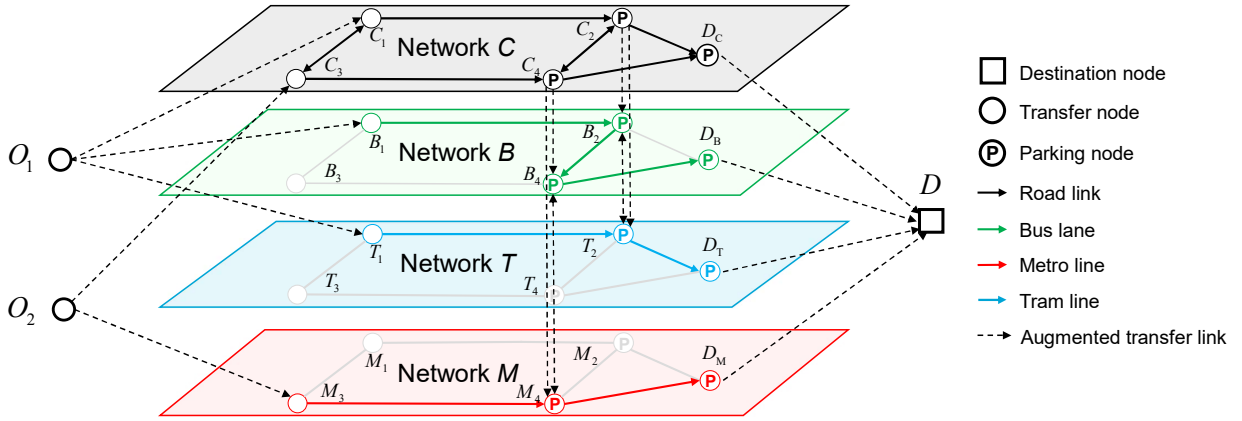
#### 17 In-vehicle link cost function

18 The cost function of in-vehicle links captures the time cost in travel between nodes within one  
 19 traffic mode and the corresponding monetary cost. The in-vehicle travel time and monetary cost  
 20 may vary according to the traffic mode, and these cost functions are defined exclusively dependent  
 21 on the link only.

22 For private car mode link, the travel time on the link can be approximated by the BPR  
 23 function, which is a monotone function of the flow on the link:

$$24 t_{a_v, \text{main}}(f_{a_v}) = t_0 \left( 1 + \alpha_1 \left( \frac{f_{a_v}}{C} \right)^{\beta_1} \right), \quad (2)$$

25 in which  $t_0$  is the free-flow travel time,  $C$  is the link capacity (veh/hour) and  $\alpha_1, \beta_1$  are parameters.



**FIGURE 3:** Augmented link-based super-network graph for a multi-modal transport network

1 The monetary cost can be calculated using the function:

$$2 \quad c_{a_v}(\mathbf{f}) = c_{\text{fuel}} l_{a_v}. \quad (3)$$

3 Note that the monetary cost can also be approximated by a monotone nonlinear function dependent  
4 on the link length and link travel time.

5 For public transit mode links, the travel time on the link can be calculated based on the link  
6 length and transit speed, assuming no congestion occurs on the transit links:

$$7 \quad t_a(\mathbf{f}) = \frac{l_a}{s_{m_t}}, m_t \in M_t. \quad (4)$$

8 The monetary cost of public transit is the transit fare, which usually consists of a base fare and  
9 a distance-based fare (see (12, 13)). In the monetary cost function, only the distance-related part is  
10 considered:

$$11 \quad c_a(\mathbf{f}) = c_{m,\text{dis}} l_a, \quad (5)$$

12 which is a function of the link attribute only. The base fare part, however, is considered in the  
13 augmented transfer link cost functions as described next. For the bus sub-network, the travel time  
14 can be different depending on the infrastructure. If there are dedicated lanes for bus, then the travel  
15 time can be estimated with equation (4). Otherwise, the bus will share the lanes with private cars,  
16 and the travel time can also be estimated with the BPR function.

### 17 *Augmented transfer link cost function*

18 In general, the transfer links between different modes can capture the access cost and egress cost.  
19 However, in this proposed method, the cost function of augmented transfer link is used to model  
20 all the cost that cannot be captured by the in-vehicle links. The cost function of each augmented  
21 transfer link is specific-defined according to its connecting traffic modes.

22 For the augmented transfer link from car mode to transit mode, e.g., links  $C_2B_2$ ,  $C_2T_2$ ,  
23  $C_4B_4$ ,  $C_4M_4$  in Figure 3, the travel time cost consists of three parts: egress time and access time  
24 from car to transit, the time needed for parking car, and the waiting time for transit. The function  
25 is given as:

$$26 \quad t_{a_t}(\mathbf{f}) = t_{a_t,\text{egress}}(\mathbf{f}) + t_{a_t,\text{access}}(\mathbf{f}) + t_{a_t,\text{park}}(\mathbf{f}) + t_{a_t,\text{wait}}(\mathbf{f}), \quad (6)$$



TABLE 1: Notations

<b>Sets</b>	Description	<b>Indices</b>	Description
$A$	Set of links	$a$	Indices of links
$A_v$	Set of in-vehicle links	$a_v$	Indices of in-vehicle links
$A_t$	Set of augmented transfer links	$a_t$	Indices of augmented transfer links
$N$	Set of nodes	$w$	Indices of OD pairs
$W$	Set of OD pairs	$p_w$	Indices of paths between $w$
$P$	Set of paths in the network	$i, j, k$	Indices of nodes
$P_w$	Set of paths between $w \in W$	$m$	Indices of modes
$\Phi$	Set of all feasible link flows	$m_t$	Indices of transit modes
$M$	Set of all traffic modes	$p$	Indices of paths
$M_t$	Set of all transit modes		
<b>Variables</b>	Description	<b>Functions</b>	Description
$\mathbf{f}$	Vector of link flows	$t_a(\mathbf{f})$	Total travel time on link $a$
$f_a$	Flow on link $a$	$t_{a,\text{access}}(\mathbf{f})$	Access time on link $a$
$\mathbf{x}$	Vector of path flows	$t_{a,\text{egress}}(\mathbf{f})$	Egress time on link $a$
$x_p$	Flow on path $p$	$t_{a,\text{wait}}(\mathbf{f})$	Waiting time on link $a$ for transit
<b>Parameters</b>	Description	$t_{a,\text{main}}(f_a)$	Main in-vehicle travel time
$d_w$	Demand of OD pair $w$	$t_{a,\text{park}}(\mathbf{f})$	Time cost on parking on link $a$
$l_{a_v}$	Length of in-vehicle link $a_v$ (km)	$c_a(\mathbf{f})$	Monetary cost on link $a$
$c_{\text{fuel}}$	Fuel/recharge cost for cars (euro/km)	$C_a(\mathbf{f})$	Total weighted cost on link $a$
$c_{\text{park}}$	Parking cost for cars (euro/h)	$C_p(\mathbf{x})$	Total travel cost on path $p$
$c_{m,\text{base}}$	Base fare of mode $m$ (euro)	$\mathbf{C}(\mathbf{f})$	The vector of all the link costs
$c_{m,\text{dis}}$	Distance-dependent fare of mode $m$ (euro/km)		
$\omega_t$	Weight on time cost		
$\omega_c$	Weight on monetary cost		
$v_{m_t}$	Frequency of transit $m_t$ (veh/h)		
$s_m$	Service speed of mode $m$ (km/h)		
$\delta_{a,p}$	Link-path incidence matrix		

1 where  $t_{a_t,\text{egress}}(\mathbf{f})$  and  $t_{a_t,\text{access}}(\mathbf{f})$  are usually assumed constant. Note that in this model, the access  
2 and egress time can be considered together, since both describe the time needed to leave from one  
3 node to the other node. For the time to parking we also use a BPR function (11):

$$4 \quad t_{a_t,\text{park}}(\mathbf{f}) = t_{0,\text{park}} \left( 1 + \alpha_2 \left( \frac{\mathbf{f}}{C_{\text{cap}}} \right)^{\beta_2} \right), \quad (7)$$

5 where  $t_{0,\text{park}}$  is the parking time when the parking lot is empty,  $C_{\text{cap}}$  is the capacity of the parking  
6 lot,  $\mathbf{f}$  refers to the links flows that share the same parking place. For example, the parking lot at  
7 node  $C_2$  accommodate cars from both transfer links  $C_2T_2$  and  $C_2B_2$ . Thus the flows on both links  
8 should be considered when calculating the parking time. Equation (7) means that the parking time

1 will increase with more cars parking in the parking lot, and the time will increase significantly  
 2 when the parking lot is approaching its maximal capacity. Meanwhile, the waiting time cost for  
 3 transit can be approximated with the function:

$$4 \quad t_{a_t, \text{wait}}(\mathbf{f}) = \frac{\alpha_3}{v_{m_t}} + \beta_3 \mathbf{f}, \quad (8)$$

5 where  $\alpha_3, \beta_3$  are calibrated parameters,  $v_{m_t}$  is the frequency of transit mode  $m_t$ , and  $\mathbf{f}$  contains  
 6 the flows that influence the waiting time, which includes the transfer flows and the prior passenger  
 7 volume before the transfer node. For example, the waiting time on transfer link  $C_2B_2$  depends not  
 8 only on the bus frequency and traffic flow on link  $C_2B_2, T_2B_2$ , but also on the passenger flow on  
 9 the in-vehicle link  $B_1B_2$ .

10 As for the monetary cost of this augmented transfer link, the cost function includes the  
 11 parking fee and the base transit fare that the passenger is transferring to, which is given as:

$$12 \quad c_{a_t}(\mathbf{f}) = c_{\text{park}}\Delta T + c_{m_t, \text{base}}, \quad (9)$$

13 where  $\Delta T$  is the time duration of parking, and  $c_{m_t, \text{base}}$  is the base part of the transit fare of mode  
 14  $m_t$  that the transfer link leads to. By encoding the parking time/monetary cost and base fare in the  
 15 transfer cost function, these costs can be automatically considered whenever the transfer link is  
 16 included in a path.

17 For other types of augmented transfer links, the cost function is defined in a similar way.  
 18 For instance, the links connecting the origins and traffic network, also known as connectors, have  
 19 a time cost function including the access time and waiting time if it links to a transit sub-network,  
 20 and have a monetary cost function including the base fare if reaching a transit node. Meanwhile,  
 21 the links approaching the destination can capture the parking time and parking fee, as well as the  
 22 access/egress time. As for the transfer links between different transit modes, the time cost function  
 23 only includes the egress/access time and waiting time, and the monetary cost equals to the base  
 24 transit fare of the subsequent transit mode. In addition, the infeasible transfer links are explicitly  
 25 excluded in the model. For example, the transfer link from transit to private car is not possible for  
 26 the network in Figure 3, and thus can be excluded during the modelling stage.

27 Note that an additional comfort cost can be added in the transfer cost function for unde-  
 28 sired transfers. For example, passengers would not prefer to transfer from one mode to another  
 29 frequently, and adding a comfort cost can model this behaviour. Therefore, the path with multiple  
 30 transfers can be excluded in the route choices through this way. Furthermore, in the illustrative ex-  
 31 ample network, only private car and several transit modes are considered. However, the model can  
 32 also be extended to accommodate a wide range of traffic modes by introducing extra layers in the  
 33 super-network, including share mobility, active mobility, and so on. There examples are not pre-  
 34 sented in detail due to limited space. In the case study, we consider an extra mode of ride-hailing,  
 35 and show the versatility of our method.

36 By defining the cost functions of all the links in the super-network in prior, all the relevant  
 37 costs are detached from the path and encoded in the corresponding individual links. These costs  
 38 are summed up automatically to calculate the total path cost for user equilibrium, without the need  
 39 to enumerate all the feasible paths. These statements are based on the assumptions given in the  
 40 next section.

#### 41 **Assumptions**

42 The following assumptions are made throughout this paper to support the main conclusions:

43 1. The path costs are assumed to be separable, which is able to approximate most of the

- 1 travel costs.
- 2 2. Only time and monetary costs are considered by the passengers. Other costs, such as
- 3 comfort cost, can also be included in the model easily.
- 4 3. The passengers have perfect rationality, and they therefore always select the route with
- 5 the lowest cost.
- 6 4. Transit operates at a fixed speed on the in-vehicle links. Other types of travel time cost
- 7 of transit can also be considered in this model.
- 8 5. The transit fare is assumed to include a base fare and a distance-related fare. Experi-
- 9 ments can be conducted to analyse the influences of non-linear fare and study the dif-
- 10 ference between two fare settings.
- 11 6. The real-world traffic networks can be described by the multi-modal transport network
- 12 as in Figure 3. This can be achieved with the pre-processing method from Najmi et al.
- 13 (14), which basically includes identifying the important nodes and links, neglecting
- 14 unnecessary information, and connecting nodes with aggregated links.
- 15 7. The travel demand between each origin–destination (OD) pair in the system is assumed
- 16 to be known and fixed. It is reasonable for strategic planning considering day-to-day
- 17 recurrent traffic scenarios.
- 18 8. Only one type of passengers is considered. This model can be extended easily to address
- 19 multiple user types that perceive different weights on the travel costs.

## 20 User equilibrium and solution algorithm

21 In this sub-section we solve the traffic assignment problem. In order to do so, the user equilibrium

22 (UE) of multi-modal transport network is formulated as a variational inequality (VI) problem,

23 based on the proposed augmented link-based super-network model, and an extra-gradient solution

24 algorithm is provided to solve the VI problem in an iterative way.

### 25 Variational inequality

26 Variational inequality has been widely used in the literature to describe user equilibrium (e.g., (15,

27 16)). According to the definition of UE, for all OD pairs  $w$ , and for  $\forall p \in P_w$ , the path flow pattern

28 is said to be in equilibrium if the following condition holds:

$$29 \quad C_p(\mathbf{f}) \begin{cases} = \lambda_w, & \text{if } x_p^* > 0, \\ \geq \lambda_w, & \text{if } x_p^* = 0, \end{cases} \quad (10)$$

30 where  $\lambda_w$  is an indicator of the least path travel cost, whose value is unknown a priori. In other

31 words, all utilized paths connecting the same OD pair have equal and minimal travel costs. Fol-

32 lowing Theorem 1 from Nagurney (17), the condition (10) can be written as the link-based VI

33 formulation:

$$34 \quad \langle \mathbf{C}(\mathbf{f}^*), (\mathbf{f} - \mathbf{f}^*) \rangle \geq 0, \quad \forall \mathbf{f} \in \Phi. \quad (11)$$

35 where  $\langle \cdot, \cdot \rangle$  denotes the inner product in Euclidean space.

36 The feasible set  $\Phi$  of link flows can be obtained according to the constraints. The problem

37 can be reformulated in terms of path-link incidence, and assuming conservation of flows at nodes.

38 First, the relationship between link flows and path flows is:

$$39 \quad f_a = \sum_{p \in P} x_p \delta_{a,p}, \quad \forall i, \forall a, \quad (12)$$

40 where  $\delta_{a,p} = 1$ , if link  $a$  is contained in path  $p$ , and 0, otherwise. In addition, the path flows should

1 satisfy the OD pair demand constraints:

$$2 \quad d_w = \sum_{p \in P_w} x_p, \quad \forall w. \quad (13)$$

3 By combining (12) and (13) and eliminating the path flow  $x_p$ , a constraint on link flow  $\mathbf{f}$  is ob-  
4 tained. In addition, extra constraints on link flows need to be considered for our model, that is,  
5 conservation of flows at nodes. For every node  $j$  in the super-network, the sum of entering flows  
6 should be equal to the sum of leaving flows:

$$7 \quad \sum_{i \in N_{j,\text{in}}} f_{a_{i,j}} = \sum_{k \in N_{j,\text{out}}} f_{a_{j,k}}, \quad \forall j \in N, \quad (14)$$

8 where  $N_{j,\text{in}}$  is the set of upstream nodes that derive a link flow to node  $j$ , while  $N_{j,\text{out}}$  is the set  
9 of downstream nodes that receive a link flow from node  $j$ . Note that when applying (14) to the  
10 origin nodes and destination nodes with taking OD pair demand into account, it is identical to the  
11 constraint (12)-(13).

### 12 *Extra-gradient algorithm solution*

13 In general, the extra-gradient method iteratively utilises two projection operators to make predic-  
14 tions and corrections until a convergence criterion is satisfied (18). More specifically, the first step  
15 is a prediction projection, and the value of the gradient in a new "extrapolated" point is used as the  
16 direction for the next step, i.e., the correction projection. Hence, this method is referred to as the  
17 double projection method. The solution procedures are given as follows:

18 **Step 0:** Initialize parameters and generate an initial solution  $\mathbf{f}^k$  by assigning all of the demand  
19 of each OD pair to the topologically least cost path, where the superscript  $k$  refers to the  
20 solution obtained in iteration  $k$ . Set the iteration counter,  $k = 0$ ; set the initial step size of  
21 the prediction projection  $\gamma$  that satisfies  $0 < \gamma < \frac{1}{L}$  where  $L$  is the Lipschitz constant of  
22 link cost function  $\mathbf{C}(\mathbf{f})$ ; set the acceptable error  $\varepsilon$ .

23 **Step 1:** Perform the prediction projection. Calculate:  $\bar{\mathbf{f}}^k = P_{\Phi}(\mathbf{f}^k - \gamma \mathbf{C}(\mathbf{f}^k))$ .

24 **Step 2:** Perform the correction projection:  $\mathbf{f}^{k+1} = P_{\Phi}(\mathbf{f}^k - \gamma \mathbf{C}(\bar{\mathbf{f}}^k))$ .

25 **Step 3:** Check if  $\|\mathbf{f}^{k+1} - \mathbf{f}^k\| \leq \varepsilon$ ; if so, stop; otherwise,  $k = k + 1$ , and go to **Step 1**.

26 Note that  $P_{\Phi}(\cdot)$  denotes the orthogonal projection onto feasible set  $\Phi$ :

$$27 \quad y = P_{\Phi}(\mathbf{f}) = \arg \min_{\forall z \in \Phi} \|\mathbf{f} - z\|. \quad (15)$$

28 Thus the calculation in Step 1 and Step 2 can be formulated as a quadratic problem. For conver-  
29 gence of the solution and equivalence of this method to the conventional path-based VI formula-  
30 tion, the interested reader can refer to (16, 18).

### 31 **CASE STUDY**

32 In this section, a numerical case study is conducted on the multi-modal transport network in Fig-  
33 ure 3, in order to validate the proposed modelling method. First, in case 1, we showed that the  
34 augmented link-based super-network model can generate exactly the same result as the conven-  
35 tional path-based, given the same network setting. Then, in case 2, we introduce an extra traffic  
36 mode, i.e., ride hailing, which makes the network more complex, and demonstrate the scalability  
37 and versatility of our method to address a more complicated network and accommodate diverse  
38 traffic modes.

- 1 **Case 1**  
2 Fixed demands for pairs  $O_1D$  and  $O_2D$  are considered, which are 800 pax/h and 1000 pax/h,  
3 respectively. This is an estimation of the morning rush hours when the passengers travel from  
4 residence to the office area, and it is assumed that the average number of passengers per car is 1,  
5 without loss of generality. The parameters of the network in Figure 3 are given in Table 2. In this  
6 case study, the buses share the same lane with private cars and have the same free flow speed, and  
7 the link lengths of the car sub-network defined in Table 2 also apply to the other sub-networks.  
8 Note that there are two values for  $c_{\text{park}}$ , in which 2.2 €/h is the parking fee at nodes  $C_2, C_4$  while  
9 the price is 5.0 €/h at node  $D_C$ . This means that parking is more expensive near the city center.  
10 All the parking lots in the network have the same capacity.

**TABLE 2:** Parameters of multi-modal network in the case study

$C$ [veh/h]	$s_{\text{car}}$ [km/h]	$s_{\text{metro}}$ [km/h]	$s_{\text{bus}}$ [km/h]	$s_{\text{tram}}$ [km/h]	$\alpha_1$	$\beta_1$	$\alpha_2$
1000	60	40	60	20	0.5	4	2.5
$d_{O_1D}$ [pax/h]	$d_{O_2D}$ [pax/h]	$l_{C_1C_2}$ [km]	$l_{C_3C_4}$ [km]	$l_{C_1C_3}$ [km]	$\beta_2$	$\alpha_3$	$\beta_3$
800	1000	10	10	6	2	0.5	0.001
$c_{\text{fuel}}$ [€/km]	$c_{\text{park}}$ [€/h]	$l_{C_2C_4}$ [km]	$l_{C_4D_c}$ [km]	$l_{C_2D_c}$ [km]	$\omega_r$	$\omega_c$	$C_{\text{cap}}$ [veh]
0.27	2.2/5.0	3	5	5	1000	10	250

- 11 The transit system setting is given in Table 3, including transit frequency and the distance-  
12 based fares for all the three transit modes.

**TABLE 3:** Public transit system setting in case 1

Metro			Bus			Tram		
$c_{m,\text{base}}$	$c_{m,\text{dis}}$	$v_{m_t}$	$c_{m,\text{base}}$	$c_{m,\text{dis}}$	$v_{m_t}$	$c_{m,\text{base}}$	$c_{m,\text{dis}}$	$v_{m_t}$
2.20 €	0.12 €	12 veh/h	2.00 €	0.10 €	6 veh/h	1.50 €	0.08 €	6 veh/h

- 12  
13 In this case, a set of feasible paths are selected in advance for each OD pair, where 5  
14 paths for  $O_1D$  and 4 paths for  $O_2D$  as shown in Table 4. Path 1,2,3,6,7 contain only one traf-  
15 fic mode, while the other paths include two modes. This implies that only the transfer links  
16  $C_2B_2, C_2T_2, C_4B_4, C_4M_4$ , i.e., from are to transit, are considered. Therefore, in the augmented link-  
17 based model, the other transfer links ( $B_2T_2, T_2B_2, B_4M_4, M_4B_4$ ) are imposed an extra inconvenience  
18 penalty cost to approximate the path-based method.

- 19 By enumerating all the paths, it is possible to formulate the path travel costs as functions  
20 of the path flows. Using the algorithm from Szeto and Jiang (9), a path-based UE can be obtained.  
21 The results of UE path flows and costs are presented in Table 5, which shows that three paths are  
22 selected for  $O_1D$  (Path 2,3,5) with the same least path cost, while two paths (Path 6,7) are chosen by  
23 the passengers from origin  $O_2$  to destination  $D$ . The UE implies that direct routes without transfers  
24 are preferred. The only transfer adopted is from private car to tram in Path 5. The corresponding  
25 link flows are presented in Table 6.

- 26 Then the UE calculation is conducted with the augmented link-based super-network model.  
27 There are 32 links in total, each of which is modelled with a link cost function. With the defined  
28 link cost function, the augmented link-based UE can be obtained using the algorithm in Section 3.5.

**TABLE 4:** The selected feasible paths for the two OD pairs

$O_1D$	Path 1	$O_1 - C_1 - C_2 - D_C - D$
	Path 2	$O_1 - B_1 - B_2 - B_4 - D_B - D$
	Path 3	$O_1 - T_1 - T_2 - D_T - D$
	Path 4	$O_1 - C_1 - C_2 - B_2 - B_4 - D_B - D$
	Path 5	$O_1 - C_1 - C_2 - T_2 - D_T - D$
$O_2D$	Path 6	$O_2 - C_3 - C_4 - D_C - D$
	Path 7	$O_2 - M_3 - M_4 - D_M - D$
	Path 8	$O_2 - C_3 - C_4 - B_4 - D_B - D$
	Path 9	$O_2 - C_3 - C_4 - M_4 - D_M - D$

**TABLE 5:** Equilibrium path flows and path costs obtained by path-based model

	Path 1	Path 2	Path 3	Path 4	Path 5	Path 6	Path 7	Path 8	Path 9
Flow [pax/h]	0	575.39	30.30	0	194.31	390.43	609.57	0	0
Cost	1108.5	1045.6	1045.6	1231.0	1045.6	1111.5	1111.5	1119.3	1123.1

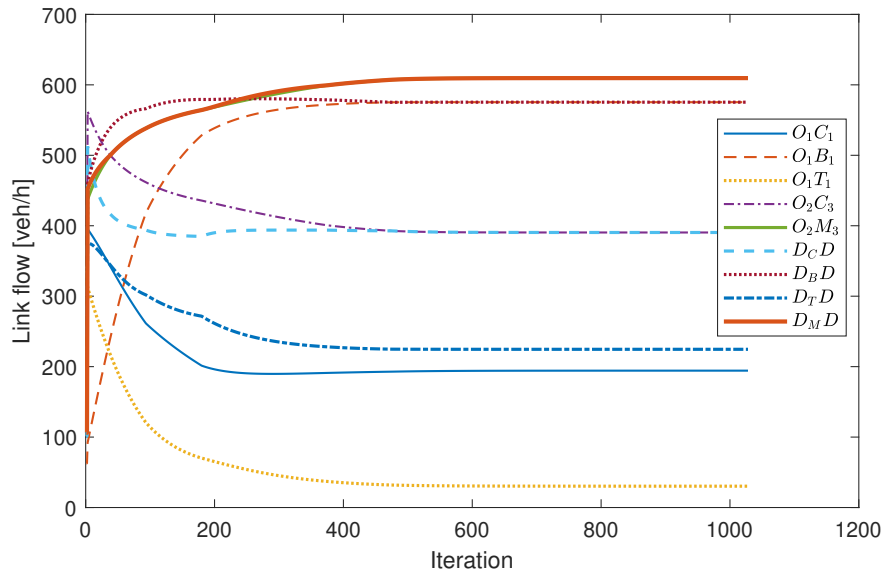
**TABLE 6:** Equilibrium link flows and link costs obtained by path-based model

Link	$O_1C_1$	$C_1C_2$	$C_1C_3$	$C_3C_1$	$C_3C_4$	$C_2C_4$	$C_4C_2$	$C_2D_C$
Flow [pax/h]	194.31	194.31	0	0	390.43	0	0	0
Link	$C_4D_C$	$D_C D$	$O_1B_1$	$B_1B_2$	$B_2B_4$	$B_4D_B$	$D_B D$	$C_2B_2$
Flow [pax/h]	390.43	390.43	575.39	575.39	575.39	575.39	575.39	0
Link	$C_4B_4$	$O_1T_1$	$T_1T_2$	$T_2D_T$	$D_T D$	$B_2T_2$	$T_2B_2$	$C_2T_2$
Flow [pax/h]	0	30.3	30.3	224.61	224.61	0	0	194.31
Link	$O_2C_3$	$O_2M_3$	$M_3M_4$	$M_4D_M$	$C_4M_4$	$B_4M_4$	$M_4B_4$	$D_M D$
Flow [pax/h]	390.43	609.57	609.57	609.57	0	0	0	609.57

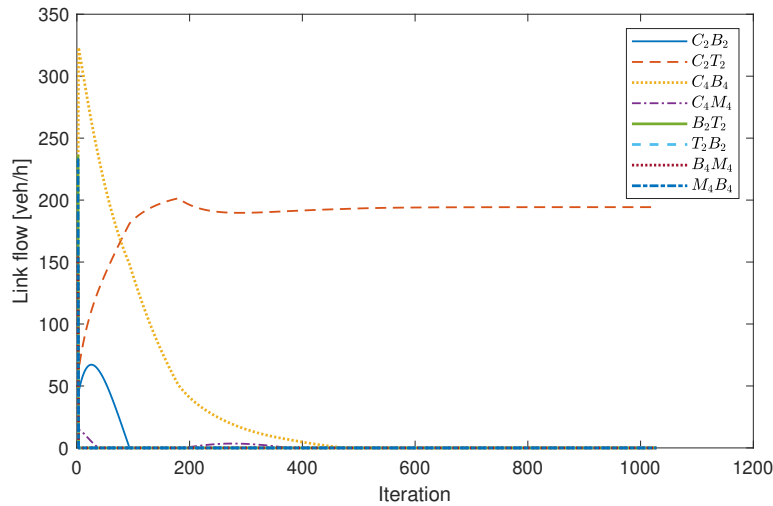
1 The iteration process of the augmented links is presented in Figure 4 and Figure 5, respectively. It  
2 is shown that the link flows converge efficiently after 400 iterations with a tolerance of  $\varepsilon = 10^{-5}$   
3 veh/h. Figure 5 shows that only the transfer link  $C_2T_2$  is selected. This is consistent with the UE  
4 result of the path-based method. The equilibrium links flows obtained with the proposed method  
5 are presented in Figure 6, which is identical to the results in Table 6 obtained by the path-based  
6 method, which validates the effectiveness of our method and the equivalence of the augmented  
7 link-based VI and the path-based VI.

## 8 Case 2

9 In Case 2, we further introduce an extra traffic mode based on the multi-modal network in Figure 3,  
10 i.e., a ride-hailing mode, as shown in Figure 7. This mode shares the same road network with the  
11 private car mode, and thus the link travel times of both modes are the same. It can be seen that  
12 the extra ride-hailing sub-network introduces more transfer links, and subsequently the feasible  
13 path set explodes compared to the original network, which make it difficult to enumerate all the



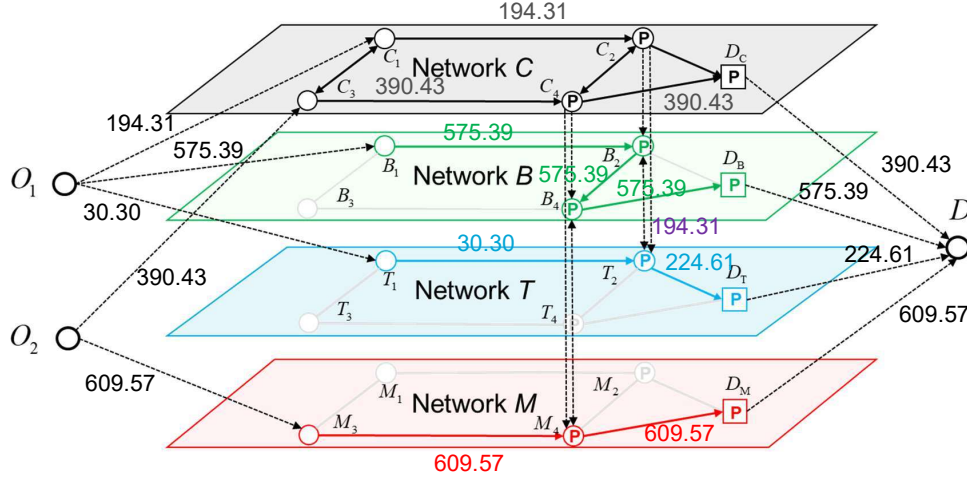
**FIGURE 4:** User equilibrium convergence process of transfer link flows between origin/destination and modes



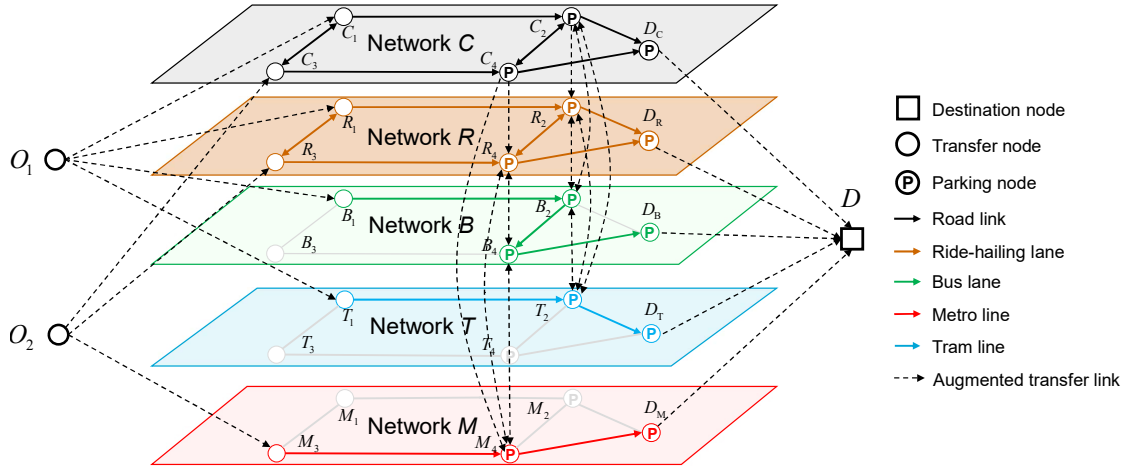
**FIGURE 5:** User equilibrium convergence process of transfer link flows between different modes

1 paths. More specifically, the number of feasible paths increases exponentially from 34 to 336,  
 2 when extending the network in Case 1 by adding one more mode. Therefore, in this case, we  
 3 will show the scalability of our model to address a more complicated network, and demonstrate its  
 4 ability to accommodate diverse traffic modes.

5 The passengers can choose to hail a ride from the origin until the destination, or transfer  
 6 to other transit modes in the middle. They can also take a taxi/Uber midway when leaving from  
 7 other modes. The cost functions of this mode can be formulated with the method introduced in the  
 8 previous section. For example, the time cost of using the ride-hailing links includes the waiting



**FIGURE 6:** User equilibrium result of the multi-modal network using the augmented link-based super-network model



**FIGURE 7:** The multi-modal transport network with a ride-hailing mode

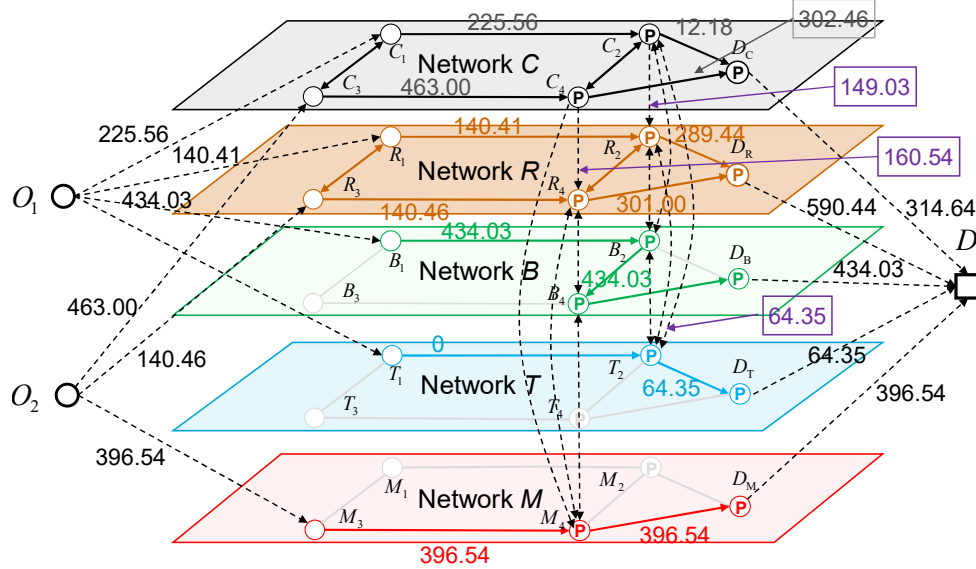
1 time for taxi and in-vehicle travelling time, while the monetary cost is only the fare. For simplicity,  
 2 the average waiting time for a taxi can be estimated by a BPR function considering the taxi service  
 3 capacity is limited ( $l$ ) as Equation (7), with  $t_0 = 0.05$  h,  $\alpha = 4$ ,  $\beta = 2$ , and  $C_{cap} = 150$  veh/h. Note  
 4 that this time cost is encoded in the cost function of the transfer link that points to the ride-hailing  
 5 mode, and  $\mathbf{f}$  are the corresponding transfer link flows that take the taxi at the same node. The  
 6 in-vehicle travel time of this mode is combined with the private car mode and bus mode, and is  
 7 included in the in-vehicle link cost function. The fare of taxi is calculated based on the function:

$$8 \quad c_a(\mathbf{f}) = c_{m,base} + c_{m,dis} l_a,$$

9 where  $c_{m,base} = 3$  € included in the corresponding transfer link, and  $c_{m,dis} = 1.2$  € included in the  
 10 in-vehicle link, which is the same as the transit mode cost function. Then the UE is calculated with  
 11 all the well-defined link cost function. The link flows of the UE results are shown in Figure 8, in  
 12 which each color corresponds to each mode and the purple numbers refer to the transfer link flows.

13





**FIGURE 8:** Link flows of user equilibrium results of the multi-modal transport network with a ride-hailing mode

1 With our method, it is only required to specify the cost of the links, which are 53 in total,  
 2 compared to 32 links of the original network. This underscores the scalability of our method. Fig-  
 3 ure 8 shows the equilibrium link flows. It can be seen that fewer passengers take public transit,  
 4 since the introduced raid-hailing mode provides a route with less travel time. Although the mone-  
 5 tary cost of ride-hailing is higher, it requires shorter travel time thanks to its higher speed and the  
 6 convenience of no parking requirements. Also notice that considerable users drive their cars and  
 7 park at node  $C_2$ , and then hail a ride to the destination. This route could save their money since they  
 8 do not need to pay the high parking fee at node  $D_C$ , and with this amount of money they can access  
 9 the destination more quickly by taking a taxi, which is also more efficient than transferring to tram  
 10 as in the previous case. Overall, taking a single mode to finish the entire trip is still preferred by  
 11 the users, e.g., the pure private car route, bus mode, and metro mode. Transfers are not favored  
 12 unless they can provide significant benefits in terms of time or money.

### 13 CONCLUSION

14 This paper develops an augmented link-based super-network method for modeling multi-modal  
 15 transport networks. This approach addresses the scalability issues of conventional multi-modal  
 16 traffic models, which require path enumeration—a process that becomes intractable in large, com-  
 17 plex networks. The proposed method introduces augmented transfer links to decouple path infor-  
 18 mation from travel cost, encoding path cost within individual links. This approach eliminates the  
 19 need for enumerating feasible paths. By explicitly modeling link costs, infeasible or undesired  
 20 transfers can be excluded a priori. Moreover, the proposed method is versatile, capable of being  
 21 applied to complex multi-modal transport networks of large size and diverse traffic modes.

22 The model is used to calculate the UE of multi-modal traffic networks, formulated as a  
 23 VI problem, which can be solved using the extra-gradient method. Case studies are conducted to  
 24 demonstrate the effectiveness of our approach. In the first case study, we validate the efficiency  
 25 and equivalence of our method compared to the path-based method using a multi-modal transport

1 network that includes multiple transit modes and a private car mode. In the second case study, an  
2 additional traffic mode is introduced, highlighting the scalability and versatility of our method.

3 This method can be further applied to real-world multi-modal transport scenarios. Addi-  
4 tionally, the generated UE results can be utilized for strategic multi-modal traffic management and  
5 network design.

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